INVENTORY MODEL FOR MANAGING DETERIORATING PRODUCT WITH STOCK LEVEL DEPENDENT DEMAND AND CONTROLLABLE DETERIORATION UNDER TRADE CREDIT AND PARTIAL BACKLOGGING

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Abstract

This paper examined a model for deteriorating items that accounts for demand that is inventory level dependent, controllable deterioration, credit facility and shortages. The paper also derived and maximized the objective function to obtain the optimal inventory management strategies. Furthermore, we presented numerical illustrations. From the numerical examples it can be observed that optimal profit and optimal quantity are sensitive to the demand rate and reduced deterioration rate parameters. We found that as the demand for the product increases, the profit also increases. Also, as the reduced deterioration rate increases, the profit decreases, which is consistent with daily business practice.

Keywords: Optimization, Optimal strategy, Trade facility, Partial backlogging, Shortages.

Introduction

Offer of trade credit practice can help to reduce inventory level since the practice encourages buyers or customers to buy more. In this regard, the work by Jui –Jung and Kuo – Nan (2010), studied deterministic inventory model that considered deteriorating items that allowed trade credit financing and capacity constraints. They assumed that the excess inventory can be stored in a rented warehouse. Furthermore, they also assumed that holding cost in the rented warehouse is more than the holding cost in the owned warehouse. They derived the objective function and maximized it to determine the optimal inventory management policies and presented numerical examples to illustrate the derived results. Bappa *et al*. (2024), considered EOQ model for deteriorating items that account for maximum lifetime, trade credit facility and partially backlogged shortages. They also presented some numerical examples and some sensitivity analyses using real-life data to gain some managerial insights. They found that depletion time depends on demand. They advanced that by halting the promotional activities during the shortage period can reduce demand.

Also, real world business practices have demonstrated that some customers are willing to wait when shortages occur during an inventory cycle while others turn to other suppliers. Thus, the research work of Khan *et al*. (2022) considered an inventory model where demand is stock-dependent for pre-payment schemes to suppliers. They also assumed that shortages are allow and the shortages are partially backlogged. Hatibaruah and Saha (2024) discussed an inventory model with ramp-type demand function and control deterioration under partial backlogging. In like manner, Jitendra (2023), also considered an inventory model that accounts for ramp-type demand function and preservative investment. Furthermore, Lesmono *et al*. (2024) also discussed an inventory models for instantaneous deteriorating items that accounts for price and stock-dependent demands, ramp-type demand.

One of the most frequent supply chain concerns is the deterioration of items during transit from a supplier's storehouse to a retailer's storehouse. In light of this, the work of Krishan *et al*., (2024) considered two-level supply chain inventory model for decaying goods was developed with two warehouse (storehouse) facilities for retailers, namely Owned Warehouse (OW) and Rented Warehouse (RW), assuming deterioration both during carrying from a supplier's storehouse to a retailer's storehouses and in the retailer's storehouses themselves. The main objective of the study is to determine the optimal ordering policy in order to maximizes the retailer's profit per unit of time.

furthermore, discussing instantaneous deteriorating items models with shortage, the work of Shah *et al.* (2024) examined an inventory model for deteriorating items where demand is stock dependent, price dependent and advertisement dependent under trade credit. Debnath *et al.* (2023) explored an inventory model for cement retailer where shortages are partially backlogged. Priyamvada *et al.* (2021) also examined a model for deteriorating items inventory model where demand function is price and stock dependent. Moreover, Lakshmi and Akilbasha (2023), work on an inventory model for deteriorating items with ramp-type demand function under partial backlogging of shortages. Similarly, Mondal *et al.* (2024) discussed an inventory model for seasonal items accounting for demand that is ramp-type and partial backlogging of shortage.

Also, discussing some other work related to the title above, the work Pathak *et al*. (2024) examined a deterministic stock model for deteriorating items that accounts for biquadratic demand function and shortages. Neeraj *el al*. (2022) work studied an inventory model that account for retention period, deterioration and backlog under two warehouses. They found that two warehouse inventory models can efficiently address various inventory problems, like depreciation, inflation, and shortages. Furthermore, the study considered how overall inventory system can be affected by shortages and cost increase. The work by Beullens (2020) considered two-echelon inventory models and discussed effect of continuous re-supply policy. The work of Sindhuja and Arathi (2023), presented an inventory model for deteriorating products that accounts for time – dependent quality demand under preservation technology. To illustrate the applicability of the model, they applied it to three

flavours of ice creams in factory, then present some numerical examples and sensitivity analyses to illustrate their results using real-time data obtained from the ice cream factory.
In practice, a lot of businesses and organizations are required to use preservation technology in their inventory management systems. This paper examined an inventory model for managing deteriorating product where demand is stock dependent, deterioration is controllable and shortages are partially backlogged.

Models Formulation

Constants and Variables

K The ordering cost per order
c the unit purchasing cost
h The holding cost per unit time (excluding interest charges)
s Unit selling price ($s > c$)
c_1 shortage cost per unit per order
c_2 Opportunity cost due to lost sales
I_e Interest earned per $ per unit of time by the retailer
I_p Interest charges per $ in stock per unit of time to the supplier
$I(t)$ The level of inventory at time t
I_m Maximum inventory level for each replenish cycle
I_b Maximum amount of demand backlogged per cycle
M Retailer's trade credit period offered by supplier per unit time
t_1 Time at which the inventory level falls to zero
T Inventory cycle length
Q Retailer's order quantity
ω Maximum capital constraint
ξ Preservation technology cost per unit time for reducing deterioration rate in order to preserve the products ($0 \leq \xi \leq \omega$)

Assumptions and Models

i. The replenishment rate is infinite
ii. Lead time is zero
iii. Planning horizon of the inventory system is infinite
iv. There is no repair or replacement of deteriorated items during the period under consideration
v. The inventory model deals with single item
vi. The reduced deterioration rate $N(\xi)$ is an increasing function of the preservation technology cost ξ where $\lim_{\xi \to \infty} N(\xi) = \theta$
vii. The demand rate function $D(t)$ is deterministic and a function of instantaneous stock level $I(t)$. When inventory is positive, $D(t)$ is given by
$D(t) = \alpha + \beta I(t), \quad 0 \leq t \leq t_1$
And when inventory is negative, $D(t)$ is given by
$D(t) = \alpha, \quad 0 \leq t \leq T$
$\alpha > 0$ and $0 \leq \beta \leq 1$ are positive constants.
viii. Shortages are allowed. The unsatisfied demand is backlogged and the fraction of shortages backordered is $b(t) = e^{-\delta t}$ $\delta > 0, t$ is the time of waiting for the next replenishment and $0 \leq b(t) \leq 1$, $b(0) = 1$. Note that if $b(x) = 1$ (or 0) for all t, then the shortages are completely backlogged (or lost). We assumed that the shortages are completely backlogged.
ix. If the trade credit period M is offered, the retailer would settle the account at $t = M$ and pay for the interest charges on the items in stock with rate I_p over the interval $[M, t_1]$ if $t_1 \geq M$ and if the retailer settles the account at $t = M$, the retailer do not need to pay any interest charge on items in stock during the whole cycle if $t_1 \leq M$
x. The retailer can accumulate revenue and earn interest from the beginning of the inventory cycle until the end of the trade credit period offered by the supplier. i.e., the retailer can accumulate revenue and earn interest during the period from $t = 0 \ to \ t = M$ with rate I_e under the trade credit conditions.

Considering the assumptions, the model for the inventory level at any given time can be described by

$$\frac{dI(t)}{dt} = \begin{cases} -\alpha - \beta I(t) - [\theta - N(\xi)]I(t) & 0 \leq t \leq t_1 \\ -\alpha b(t) & t_1 \leq t \leq T \end{cases} \quad (1)$$

With boundary condition $I(t_1) = 0$
During the interval $[0, t_1]$, the inventory level decreases due to combined effects of deterioration and demand, and the inventory drops to zero during the time interval $[0, t_1]$. During the interval $[t_1, T]$, shortages occurred which are completely backlogged. From equation (1), the rate of change of inventory level at any time t can be represented by the following differential equation:

$$\frac{dI(t)}{dt} = -\alpha - \beta I(t) - [\theta - N(\xi)]I(t) \quad 0 \leq t \leq t_1$$

$$\frac{dI_1}{dt} = -\alpha b(t) \quad t_1 \leq t \leq T$$

With the boundary conditions $I(0) = I_M, I_1(t_1) = 0$

$$\frac{dI(t)}{dt} = -\alpha - \beta I(t) - [\theta - N(\xi)]I(t) \quad 0 \leq t \leq t_1$$

$$\frac{dI(t)}{dt} = -\alpha - [\beta + (\theta - N(\xi))]I(t)$$
$$= -(\alpha + [\beta + \theta - N(\xi)]I(t))$$

$$\int_t^{t_1} \frac{dI(t)}{\alpha+[\beta+\theta-N(\xi)]I(t)} = \int_t^{t_1} -dt$$
$$\int_t^{t_1} \frac{dI(t)}{\alpha+[\beta+\theta-N(\xi)]I(t)} = -\int_t^{t_1} dt$$
$$\ln\left[\frac{\alpha+[\beta+\theta-N(\xi)]I(t_1)}{\alpha+[\beta+\theta-N(\xi)]I(t)}\right] = -(\beta+\theta-N(\xi))(t_1-t)$$
$$\frac{\alpha+[\beta+\theta-N(\xi)]I(t_1)}{\alpha+[\beta+\theta-N(\xi)]I(t)} = e^{-(\beta+\theta-N(\xi))(t_1-t)}$$
Since $I(t_1)=0$, we Have
$$\frac{\alpha}{\alpha+[\beta+\theta-N(\xi)]I(t)} = e^{-(\beta+\theta-N(\xi))(t_1-t)}$$
$$\alpha+[\beta+\theta-N(\xi)]I(t) = \alpha e^{(\beta+\theta-N(\xi))(t_1-t)}$$
$$[\beta+\theta-N(\xi)]I(t) = \alpha e^{(\beta+\theta-N(\xi))(t_1-t)} - \alpha$$
$$I(t) = \frac{\alpha[e^{(\beta+\theta-N(\xi))(t_1-t)}-1]}{\beta+\theta-N(\xi)}$$
$$I_m = I(0) = \frac{\alpha[e^{(\beta+\theta-N(\xi))t_1}-1]}{\beta+\theta-N(\xi)}$$

The Derivation of the Components of the Objective Function

Ordering cost per cycle $= K$

Holding Cost $= h\int_0^{t_1} I(t)\,dt$
$$= h\int_0^{t_1} \frac{\alpha[e^{(\beta+\theta-N(\xi))(t_1-t)}-1]}{\beta+\theta-N(\xi)}dt$$
$$= \frac{\alpha h}{\beta+\theta-N(\xi)}\int_0^{t_1}[e^{(\beta+\theta-N(\xi))(t_1-t)}-1]\,dt$$
$$= \frac{\alpha h}{\beta+\theta-N(\xi)}\left|\frac{e^{(\beta+\theta-N(\xi))(t_1-t)}}{-(\beta+\theta-N(\xi))} - t\right|_0^{t_1}$$
$$= \frac{\alpha h}{\beta+\theta-N(\xi)}\left(\left[\frac{e^{(\beta+\theta-N(\xi))(t_1-t_1)}}{-(\beta+\theta-N(\xi))} - t_1\right] - \left[\frac{e^{(\beta+\theta-N(\xi))(t_1-0)}}{-(\beta+\theta-N(\xi))} - 0\right]\right)$$
$$= \frac{\alpha h}{\beta+\theta-N(\xi)}\left(-\frac{1}{(\beta+\theta-N(\xi))} - t_1 + \frac{e^{(\beta+\theta-N(\xi))t_1}}{(\beta+\theta-N(\xi))}\right)$$
$$= \frac{\alpha h}{\beta+\theta-N(\xi)}\left(\frac{[e^{(\beta+\theta-N(\xi))t_1}-1]}{\beta+\theta-N(\xi)} - t_1\right)$$

For unsatisfied demand partially backlogged, we have
$$\frac{dI_1}{dt} = -\alpha b(t) \qquad t_1 \le t \le T$$
$b(t) = e^{-\delta t}$, therefore,
$$\frac{dI_1}{dt} = -\alpha e^{-\delta t}$$
$$I_1(t) = -\alpha\int_{t_1}^{t} e^{-\delta s}ds = -\alpha \times \frac{1}{-\delta}e^{-\delta s}\Big|_{t_1}^{t}$$
$$= \frac{\alpha}{\delta}e^{-\delta s}\Big|_{t_1}^{t} = \frac{\alpha}{\delta}e^{-\delta t} - \frac{\alpha}{\delta}e^{-\delta t_1}$$
$$= \frac{\alpha}{\delta}\left(e^{-\delta t} - e^{-\delta t_1}\right)$$

Maximum backordered quantity (maximum amount of demand backlogged)
$$I_1(T) = I_{1m} = \frac{\alpha}{\delta}\left(e^{-\delta T} - e^{-\delta t_1}\right)$$

Lost sales or opportunity cost $= -c_2\alpha\int_{t_1}^{T}(1-e^{-\delta t})dt = -c_2\alpha\left(t+\frac{e^{-\delta t}}{\delta}\right)\Big|_{t_1}^{T}$
$$= -c_2\alpha\left[\left(T+\frac{e^{-\delta T}}{\delta}\right) - \left(t_1+\frac{e^{-\delta t_1}}{\delta}\right)\right]$$
$$= c_2\alpha\left[t_1 + \frac{e^{-\delta t_1}}{\delta} - T - \frac{e^{-\delta T}}{\delta}\right]$$

Purchase cost:

Maximum quantity $= Q = I_m - I_{1m}$

$$Q = I_m - I_{1m} = \frac{\alpha[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta+\theta-N(\xi)} - \frac{\alpha}{\delta}\left(e^{-\delta T} - e^{-\delta t_1}\right)$$
$$= \frac{\alpha[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta+\theta-N(\xi)} + \frac{\alpha}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right)$$

$$\text{Purchase cost} = cQ = c\left(\frac{\alpha[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta+\theta-N(\xi)} + \frac{\alpha}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right)\right)$$
$$= \frac{\alpha c[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta+\theta-N(\xi)} + \frac{\alpha c}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right)$$

Shortage cost:

$$\text{Shortage cost} = -c_1\int_{t_1}^{T} I_1(t)\,dt = \frac{-c_1\alpha}{\delta}\int_{t_1}^{T}\left(e^{-\delta t} - e^{-\delta t_1}\right)dt$$
$$= \frac{-c_1\alpha}{\delta}\left(-\frac{e^{-\delta t}}{\delta} - te^{-\delta t_1}\right)\Big|_{t_1}^{T}$$
$$= \frac{-c_1\alpha}{\delta}\left[\left(-\frac{e^{-\delta T}}{\delta} - Te^{-\delta T}\right) - \left(-\frac{e^{-\delta t_1}}{\delta} - t_1e^{-\delta t_1}\right)\right]$$
$$= \frac{-c_1\alpha}{\delta}\left[-\frac{e^{-\delta T}}{\delta} - Te^{-\delta T} + \frac{e^{-\delta t_1}}{\delta} + t_1e^{-\delta t_1}\right]$$
$$= \frac{c_1\alpha}{\delta}\left[\frac{e^{-\delta T}}{\delta} + Te^{-\delta T} - \frac{e^{-\delta t_1}}{\delta} - t_1e^{-\delta t_1}\right]$$

Preservation technology cost $= \xi T$

Sales Revenue**:**

$$\text{Sales revenue} = s\left(\int_0^{t_1} D(t)\,dt - I_1(T)\right)$$
$$= s\int_0^{t_1}(\alpha + \beta I(t))\,dt - \frac{s\alpha}{\delta}\left(e^{-\delta T} - e^{-\delta t_1}\right)$$
$$= s\int_0^{t_1}\left(\alpha + \beta\frac{\alpha[e^{(\beta+\theta-N(\xi))(t_1-t)} - 1]}{\beta+\theta-N(\xi)}\right)dt + \frac{s\alpha}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right)$$
$$= s\int_0^{t_1}\left(\alpha + \beta\alpha\frac{e^{(\beta+\theta-N(\xi))(t_1-t)}}{\beta+\theta-N(\xi)} - \frac{\beta\alpha}{\beta+\theta-N(\xi)}\right)dt + + \frac{s\alpha}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right)$$
$$= s\alpha\left[\frac{t_1[\theta-N(\xi)]}{\beta+\theta-N(\xi)} + \frac{\beta(e^{(\beta+\theta-N(\xi))t_1} - 1)}{[\beta+\theta-N(\xi)]^2}\right] + \frac{s\alpha}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right)$$

$$\text{Deteriorating cost} = d_c\left[I_m - \int_0^{t_1} D(t)\,dt\right] = d_c\left[I_m - \int_0^{t_1}(\alpha + \beta I(t))\,dt\right]$$
$$= d_c\left[\frac{\alpha[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta+\theta-N(\xi)} - \int_0^{t_1}\left(\alpha + \beta\frac{\alpha[e^{(\beta+\theta-N(\xi))(t_1-t)} - 1]}{\beta+\theta-N(\xi)}\right)dt\right]$$
$$= d_c\left[\frac{\alpha[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta+\theta-N(\xi)} - \left(\frac{\alpha t_1[\theta-N(\xi)]}{\beta+\theta-N(\xi)} + \frac{\alpha\beta(e^{(\beta+\theta-N(\xi))t_1} - 1)}{[\beta+\theta-N(\xi)]^2}\right)\right]$$
$$= d_c\left[\frac{\alpha[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta+\theta-N(\xi)} - \frac{\alpha t_1[\theta-N(\xi)]}{\beta+\theta-N(\xi)} - \frac{\alpha\beta(e^{(\beta+\theta-N(\xi))t_1} - 1)}{[\beta+\theta-N(\xi)]^2}\right]$$
$$= d_c\left[\frac{\alpha[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta+\theta-N(\xi)} - \frac{\alpha\beta(e^{(\beta+\theta-N(\xi))t_1} - 1)}{[\beta+\theta-N(\xi)]^2} - \frac{\alpha t_1[\theta-N(\xi)]}{\beta+\theta-N(\xi)}\right]$$
$$= d_c\left[\frac{\alpha[\beta+\theta-N(\xi)][e^{(\beta+\theta-N(\xi))t_1} - 1] - \alpha\beta(e^{(\beta+\theta-N(\xi))t_1} - 1)}{[\beta+\theta-N(\xi)]^2} - \frac{\alpha t_1[\theta-N(\xi)]}{\beta+\theta-N(\xi)}\right]$$
$$= d_c\left[\frac{\alpha[\theta-N(\xi)][e^{(\beta+\theta-N(\xi))t_1} - 1]}{[\beta+\theta-N(\xi)]^2} - \frac{\alpha t_1[\theta-N(\xi)]}{\beta+\theta-N(\xi)}\right]$$
$$= \frac{d_c\alpha[\theta-N(\xi)]}{\beta+\theta-N(\xi)}\left[\frac{[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta+\theta-N(\xi)} - t_1\right]$$

Interest payable and earned interest:

If the end of the credit period is shorter than or equal to the length of period in which the inventory is positive $(M \leq t_1)$, payment for goods is settled and the retailer starts paying the capital opportunity cost for the items in stock with rate I_p. We also assumed

that while the account is yet to be settled, the retailer can sell the goods and continue to accumulate sales revenue and earn interest with the rate I_e. Hence, interest earned and payable per cycle for different cases are given below:
Case I: $M \leq t_1$

$$\text{interest payable} = cI_p \int_M^{t_1} I(t)dt$$

$$= cI_p \int_m^{t_1} \left(\frac{\alpha[e^{(\beta+\theta-N(\xi))(t_1-t)} - 1]}{\beta+\theta-N(\xi)} \right) dt$$

$$= \frac{cI_p\alpha}{\beta+\theta-N(\xi)} \int_M^{t_1} \left(e^{(\beta+\theta-N(\xi))(t_1-t)} - 1 \right) dt$$

$$= \frac{cI_p\alpha}{\beta+\theta-N(\xi)} \left(\frac{e^{(\beta+\theta-N(\xi))(t_1-t)}}{-(\beta+\theta-N(\xi))} - t \right) \Big|_M^{t_1}$$

$$= \frac{cI_p\alpha}{\beta+\theta-N(\xi)} \left[\left(\frac{e^{(\beta+\theta-N(\xi))(t_1-t_1)}}{-(\beta+\theta-N(\xi))} - t_1 \right) - \left(\frac{e^{(\beta+\theta-N(\xi))(t_1-M)}}{-(\beta+\theta-N(\xi))} - M \right) \right]$$

$$= \frac{cI_p\alpha}{\beta+\theta-N(\xi)} \left[\frac{e^{(\beta+\theta-N(\xi))(t_1-M)}}{(\beta+\theta-N(\xi))} + M - \frac{1}{\beta+\theta-N(\xi)} - t_1 \right]$$

$$= \frac{cI_p\alpha}{\beta+\theta-N(\xi)} \left[\frac{e^{(\beta+\theta-N(\xi))(t_1-M)} - 1}{(\beta+\theta-N(\xi))} + M - t_1 \right]$$

$$\text{interest earned} = sI_e \int_0^m \int_0^t D(s)ds\, dt$$

$$= sI_e \int_0^m \left[\int_0^t D(s)ds \right] dt$$

$$= sI_e \int_0^m \left[\int_0^t (\alpha + \beta I(s))ds \right] dt$$

$$= sI_e \int_0^m \left[\int_0^t \left(\alpha + \frac{\beta\alpha[e^{(\beta+\theta-N(\xi))(t_1-s)} - 1]}{\beta+\theta-N(\xi)} \right) ds \right] dt$$

$$= sI_e \int_0^M \left[\int_0^t \left(\alpha + \frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-s)}}{\beta+\theta-N(\xi)} - \frac{\beta\alpha}{\beta+\theta-N(\xi)} \right) ds \right] dt$$

$$= sI_e \int_0^M \left[\left(\alpha s - \frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-s)}}{[\beta+\theta-N(\xi)]^2} - \frac{\beta\alpha s}{\beta+\theta-N(\xi)} \right) \Big|_0^t \right] dt$$

$$= sI_e \int_0^M \left[\left(\alpha t - \frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-t)}}{[\beta+\theta-N(\xi)]^2} - \frac{\beta\alpha t}{\beta+\theta-N(\xi)} \right) - \left(0 - \frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-0)}}{[\beta+\theta-N(\xi)]^2} - 0 \right) \right] dt$$

$$= sI_e \int_0^M \left[\alpha t - \frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-t)}}{[\beta+\theta-N(\xi)]^2} - \frac{\beta\alpha t}{\beta+\theta-N(\xi)} + \frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2} \right] dt$$

$$= sI_e \left(\frac{\alpha t^2}{2} + \frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-t)}}{[\beta+\theta-N(\xi)]^3} - \frac{\beta\alpha t^2}{2(\beta+\theta-N(\xi))} + \frac{\beta\alpha t e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2} \right) \Big|_0^M$$

$$= sI_e \left[\left(\frac{\alpha M^2}{2} + \frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-M)}}{[\beta+\theta-N(\xi)]^3} - \frac{\beta\alpha M^2}{2(\beta+\theta-N(\xi))} + \frac{\beta\alpha M e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2} \right) - \left(0 + \frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^3} - 0 + 0 \right) \right]$$

$$= sI_e \left[\frac{\alpha M^2}{2} + \frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-M)}}{[\beta+\theta-N(\xi)]^3} - \frac{\beta\alpha M^2}{2(\beta+\theta-N(\xi))} + \frac{\beta\alpha M e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2} - \frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^3} \right]$$

$$= sI_e \left[\frac{\alpha M^2}{2} - \frac{\beta\alpha M^2}{2(\beta+\theta-N(\xi))} + \frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-M)}}{[\beta+\theta-N(\xi)]^3} - \frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^3} + \frac{\beta\alpha M e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2} \right]$$

$$= sI_e \left[\frac{\alpha M^2(\beta+\theta-N(\xi)) - \beta\alpha M^2}{2(\beta+\theta-N(\xi))} + \frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}(e^{-(\beta+\theta-N(\xi))M} - 1)}{[\beta+\theta-N(\xi)]^3} \right.$$

$$+\frac{\beta\alpha M e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\Bigg]$$

$$= sI_e\left[\frac{\alpha M^2(\theta-N(\xi))}{2(\beta+\theta-N(\xi))}+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}(e^{-(\beta+\theta-N(\xi))M}-1)}{[\beta+\theta-N(\xi)]^3}+\frac{\beta\alpha M e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right]$$

Case II: $t_1 \leq M$

Here, the cycle time t, is less than or equal to the credit period M, so no interest will be paid. Hence, the retailer pays no interest at the end of the inventory cycle. Thus,

Interest payable $= 0$

Thus, from time 0 to t_1, the retailer can sell the goods and continue to accumulate sales revenue to earn interest $sI_e \int_0^{t_1}\int_0^t D(s)\,ds\,dt$. Also, from time t_1 to M, the retailer uses the sales revenue generated in $[0, t_1]$ to earn interest $sI_e \int_0^{t_1} D(s)ds\,(M-t_1)$. Thus, the interest earned in this period per cycle can be described by:

$$\text{Interest earned} = sI_e\left[\int_0^{t_1}\int_0^t D(s)\,ds\,dt+\int_0^{t_1} D(t)dt\,(M-t_1)\right]$$

$$= sI_e\int_0^{t_1}\int_0^t D(s)\,ds\,dt + sI_e\int_0^{t_1} D(t)dt\,(M-t_1)$$

$$= sI_e\int_0^{t_1}\int_0^t D(s)\,ds\,dt + sI_e\int_0^{t_1} D(t)dt\,(M-t_1)$$

$$= sI_e\int_0^{t_1}\left[\int_0^t D(s)ds\right]dt + sI_e\int_0^{t_1} D(t)dt\,(M-t_1)$$

$$= sI_e\int_0^{t_1}\left[\int_0^t(\alpha+\beta I(s))ds\right]dt + sI_e(M-t_1)\int_0^{t_1}(\alpha+\beta I(t))dt$$

$$= sI_e\int_0^{t_1}\left[\int_0^t\left(\alpha+\frac{\beta\alpha[e^{(\beta+\theta-N(\xi))(t_1-s)}-1]}{\beta+\theta-N(\xi)}\right)ds\right]dt$$

$$+sI_e(M-t_1)\int_0^{t_1}\left(\alpha+\frac{\beta\alpha[e^{(\beta+\theta-N(\xi))(t_1-t)}-1]}{\beta+\theta-N(\xi)}\right)dt$$

$$= sI_e\int_0^{t_1}\left[\int_0^t\left(\alpha+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-s)}}{\beta+\theta-N(\xi)}-\frac{\beta\alpha}{\beta+\theta-N(\xi)}\right)ds\right]dt$$

$$+sI_e(M-t_1)\int_0^{t_1}\left(\alpha+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-s)}}{\beta+\theta-N(\xi)}-\frac{\beta\alpha}{\beta+\theta-N(\xi)}\right)dt$$

$$= sI_e\int_0^{t_1}\left[\left(\alpha s-\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-s)}}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha s}{\beta+\theta-N(\xi)}\right)\Big|_0^t\right]dt$$

$$+sI_e(M-t_1)\left(\alpha t-\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-t)}}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha t}{\beta+\theta-N(\xi)}\right)\Big|_0^{t_1}$$

$$= sI_e\int_0^{t_1}\left[\left(\alpha t-\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-t)}}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha t}{\beta+\theta-N(\xi)}\right)-\left(0-\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-0)}}{[\beta+\theta-N(\xi)]^2}-0\right)\right]dt$$

$$+sI_e(M-t_1)\left[\left(\alpha t_1-\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-t_1)}}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha t_1}{\beta+\theta-N(\xi)}\right)-\left(0-\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-0)}}{[\beta+\theta-N(\xi)]^2}-0\right)\right]$$

$$= sI_e\int_0^{t_1}\left[\alpha t-\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-t)}}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha t}{\beta+\theta-N(\xi)}+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right]dt$$

$$+sI_e(M-t_1)\left[\alpha t_1-\frac{\beta\alpha}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha t_1}{\beta+\theta-N(\xi)}+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right]$$

$$= sI_e\left(\frac{\alpha t}{2}+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-t)}}{[\beta+\theta-N(\xi)]^3}-\frac{\beta\alpha t}{2(\beta+\theta-N(\xi))}+\frac{\beta\alpha t e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right)\Big|_0^{t_1}$$

$$+sI_e(M-t_1)\left[\alpha t_1-\frac{\beta\alpha}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha t_1}{\beta+\theta-N(\xi)}+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right]$$

$$= sI_e\left[\left(\frac{\alpha t_1^2}{2}+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-t_1)}}{[\beta+\theta-N(\xi)]^3}-\frac{\beta\alpha t_1^2}{2(\beta+\theta-N(\xi))}+\frac{\beta\alpha t_1 e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right)\right.$$

$$\left.-\left(0+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))(t_1-0)}}{[\beta+\theta-N(\xi)]^3}-0+0\right)\right]$$

$$+sI_e(M-t_1)\left[\alpha t_1-\frac{\beta\alpha}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha t_1}{\beta+\theta-N(\xi)}+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right]$$

$$=sI_e\left[\frac{\alpha t_1^2}{2}+\frac{\beta\alpha}{[\beta+\theta-N(\xi)]^3}-\frac{\beta\alpha t_1^2}{2(\beta+\theta-N(\xi))}+\frac{\beta\alpha t_1 e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^3}\right]$$

$$+sI_e(M-t_1)\left[\alpha t_1-\frac{\beta\alpha}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha t_1}{\beta+\theta-N(\xi)}+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right]$$

$$=sI_e\left[\frac{\alpha t_1^2}{2}-\frac{\beta\alpha t_1^2}{2(\beta+\theta-N(\xi))}-\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^3}+\frac{\beta\alpha}{[\beta+\theta-N(\xi)]^3}+\frac{\beta\alpha t_1 e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right]$$

$$+sI_e(M-t_1)\left[\alpha t_1-\frac{\beta\alpha t_1}{\beta+\theta-N(\xi)}+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha}{[\beta+\theta-N(\xi)]^2}\right]$$

$$=sI_e\left[\frac{\alpha t_1^2(\beta+\theta-N(\xi))-\beta\alpha t_1^2}{2(\beta+\theta-N(\xi))}-\frac{\beta\alpha[e^{(\beta+\theta-N(\xi))t_1}-1]}{[\beta+\theta-N(\xi)]^3}+\frac{\beta\alpha t_1 e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right]$$

$$+sI_e(M-t_1)\left[\frac{\alpha t_1[\beta+\theta-N(\xi)]-\beta\alpha t_1}{\beta+\theta-N(\xi)}+\frac{\beta\alpha[e^{(\beta+\theta-N(\xi))t_1}-1]}{[\beta+\theta-N(\xi)]^2}\right]$$

$$=sI_e\left[\frac{\alpha t_1^2(\theta-N(\xi))}{2(\beta+\theta-N(\xi))}+\frac{\beta\alpha t_1 e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha[e^{(\beta+\theta-N(\xi))t_1}-1]}{[\beta+\theta-N(\xi)]^3}\right]$$

$$+sI_e(M-t_1)\left[\frac{\alpha t_1[\theta-N(\xi)]}{\beta+\theta-N(\xi)}+\frac{\beta\alpha[e^{(\beta+\theta-N(\xi))t_1}-1]}{[\beta+\theta-N(\xi)]^2}\right]$$

Profit maximization Problem Under Partial Backlogging of Shortages

Our problem here is to determine the optimal value of t which optimizes $P(t)$. The necessary and the sufficient conditions for maximization of the total profit function $P(t)$ are:

$$\frac{d}{dt}P(t)=0 \qquad (16)$$

Equation (16) can be solved for t to obtain the optimal value of t (say t^*). The sufficient condition for $P(t)$ to be a maximum is that the

$$\frac{d^2}{dt^2}P(t)<0 \qquad (17)$$

If we put the above components into consideration, then the profit function for the retailer can be described by:

$$P_1(t_1)=\frac{1}{T}\left\{\begin{matrix}\text{sales revenue + interest earned − odering cost − holding cost − purchasing cost}\\ \text{−deteriorating cost − shortage cost − opportunity cost − preservation cost}\\ \text{−interest payable}\end{matrix}\right\}$$

Case I: $M\le t_1$

Here, opportunity cost $\neq 0$, therefore,

$$P_1(t_1)=\frac{1}{T}\Bigg\{s\alpha\left[\frac{t_1[\theta-N(\xi)]}{\beta+\theta-N(\xi)}+\frac{\beta(e^{(\beta+\theta-N(\xi))t_1}-1)}{[\beta+\theta-N(\xi)]^2}\right]+\frac{s\alpha}{\delta}\left(e^{-\delta t_1}-e^{-\delta T}\right)+sI_e\left[\frac{\alpha M^2(\theta-N(\xi))}{2(\beta+\theta-N(\xi))}+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}(e^{-(\beta+\theta-N(\xi))M}-1)}{[\beta+\theta-N(\xi)]^3}+\frac{\beta\alpha M e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right]-K$$
$$-\frac{\alpha h}{\beta+\theta-N(\xi)}\left(\frac{[e^{(\beta+\theta-N(\xi))t_1}-1]}{\beta+\theta-N(\xi)}-t_1\right)-\left(\frac{\alpha c[e^{(\beta+\theta-N(\xi))t_1}-1]}{\beta+\theta-N(\xi)}+\frac{\alpha c}{\delta}\left(e^{-\delta t_1}-e^{-\delta T}\right)\right)$$
$$-\frac{d_c\alpha[\theta-N(\xi)]}{\beta+\theta-N(\xi)}\left[\frac{[e^{(\beta+\theta-N(\xi))t_1}-1]}{\beta+\theta-N(\xi)}-t_1\right]-\frac{c_1\alpha}{\delta}\left[\frac{e^{-\delta T}}{\delta}+Te^{-\delta T}-\frac{e^{-\delta t_1}}{\delta}-t_1e^{-\delta t_1}\right]$$
$$-c_2\alpha\left[t_1+\frac{e^{-\delta t_1}}{\delta}-T-\frac{e^{-\delta T}}{\delta}\right]-\xi T-\frac{cI_p\alpha}{\beta+\theta-N(\xi)}\left[\frac{e^{(\beta+\theta-N(\xi))(t_1-M)}-1}{(\beta+\theta-N(\xi))}+M-t_1\right]\Bigg\} \qquad (18)$$

Proposition I. if $\beta=0$, $s\delta+c_1t_1-c\delta-c_1-c_2\delta<0$, $e^{-(\beta+\theta-N(\xi))M}\le 1$ and $s\beta+sI_e\beta M-c(\beta+\theta-N(\xi))-h-d_c[\theta-N(\xi)]<0$, then $\frac{d^2}{dt^2}P_1(t)<0$ and (t_1^*) is a maximum solution of Equ. (18).

Proof:

$$\frac{dP_1(t_1)}{dt_1}=\frac{1}{T}\left\{s\alpha\left[\frac{[\theta-N(\xi)]}{\beta+\theta-N(\xi)}+\frac{\beta e^{(\beta+\theta-N(\xi))t_1}}{\beta+\theta-N(\xi)}\right]+\frac{s\alpha}{\delta}\left(-\delta e^{-\delta t_1}\right)\right.$$
$$+sI_e\left[\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}\left(e^{-(\beta+\theta-N(\xi))M}-1\right)}{[\beta+\theta-N(\xi)]^2}+\frac{\beta\alpha Me^{(\beta+\theta-N(\xi))t_1}}{\beta+\theta-N(\xi)}\right]$$
$$-\frac{\alpha h}{\beta+\theta-N(\xi)}\left(e^{(\beta+\theta-N(\xi))t_1}-1\right)-\left(\alpha ce^{(\beta+\theta-N(\xi))t_1}+\frac{\alpha c}{\delta}\left(-\delta e^{-\delta t_1}\right)\right)$$
$$-\frac{d_c\alpha[\theta-N(\xi)]}{\beta+\theta-N(\xi)}\left[e^{(\beta+\theta-N(\xi))t_1}-1\right]-\frac{c_1\alpha}{\delta}\left[e^{-\delta t_1}-\left(e^{-\delta t_1}-t_1\delta e^{-\delta t_1}\right)\right]$$
$$\left.-c_2\alpha\left[1-e^{-\delta t_1}\right]-\frac{cI_p\alpha}{\beta+\theta-N(\xi)}\left[e^{(\beta+\theta-N(\xi))(t_1-M)}-1\right]\right\}$$
$$=\frac{1}{T}\left\{s\alpha\left[\frac{[\theta-N(\xi)]}{\beta+\theta-N(\xi)}+\frac{\beta e^{(\beta+\theta-N(\xi))t_1}}{\beta+\theta-N(\xi)}\right]-s\alpha e^{-\delta t_1}\right.$$
$$+sI_e\left[\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}\left(e^{-(\beta+\theta-N(\xi))M}-1\right)}{[\beta+\theta-N(\xi)]^2}+\frac{\beta\alpha Me^{(\beta+\theta-N(\xi))t_1}}{\beta+\theta-N(\xi)}\right]$$
$$-\frac{\alpha h}{\beta+\theta-N(\xi)}\left(e^{(\beta+\theta-N(\xi))t_1}-1\right)-\left(\alpha ce^{(\beta+\theta-N(\xi))t_1}-\alpha ce^{-\delta t_1}\right)$$
$$-\frac{d_c\alpha[\theta-N(\xi)]}{\beta+\theta-N(\xi)}\left[e^{(\beta+\theta-N(\xi))t_1}-1\right]-c_1\alpha t_1e^{-\delta t_1}-c_2\alpha\left[1-e^{-\delta t_1}\right]$$
$$\left.-\frac{cI_p\alpha}{\beta+\theta-N(\xi)}\left[e^{(\beta+\theta-N(\xi))(t_1-M)}-1\right]\right\}$$

$$\frac{d^2P_1(t_1)}{dt^2}=\frac{1}{T}\left\{s\alpha\beta e^{(\beta+\theta-N(\xi))t_1}+s\alpha\delta e^{-\delta t_1}\right.$$
$$+sI_e\left[\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}\left(e^{-(\beta+\theta-N(\xi))M}-1\right)}{\beta+\theta-N(\xi)}+\beta\alpha Me^{(\beta+\theta-N(\xi))t_1}\right]$$
$$-\alpha he^{(\beta+\theta-N(\xi))t_1}-\alpha c\left(\beta+\theta-N(\xi)\right)e^{(\beta+\theta-N(\xi))t_1}-\alpha c\delta e^{-\delta t_1}$$
$$-d_c\alpha[\theta-N(\xi)]e^{(\beta+\theta-N(\xi))t_1}-c_1\alpha e^{-\delta t_1}+c_1\alpha\delta t_1e^{-\delta t_1}-c_2\alpha\delta e^{-\delta t_1}$$
$$\left.-cI_p\alpha e^{(\beta+\theta-N(\xi))(t_1-M)}\right\}$$
$$=\frac{1}{T}\left\{s\alpha\delta e^{-\delta t_1}+c_1\alpha\delta t_1e^{-\delta t_1}-\alpha c\delta e^{-\delta t_1}-c_1\alpha e^{-\delta t_1}-c_2\alpha\delta e^{-\delta t_1}+s\alpha\beta e^{(\beta+\theta-N(\xi))t_1}\right.$$
$$+sI_e\beta\alpha Me^{(\beta+\theta-N(\xi))t_1}-\alpha c(\beta+\theta-N(\xi))e^{(\beta+\theta-N(\xi))t_1}-\alpha he^{(\beta+\theta-N(\xi))t_1}$$
$$-d_c\alpha[\theta-N(\xi)]e^{(\beta+\theta-N(\xi))t_1}+\frac{sI_e\beta\alpha e^{(\beta+\theta-N(\xi))t_1}\left(e^{-(\beta+\theta-N(\xi))M}-1\right)}{\beta+\theta-N(\xi)}$$
$$\left.-cI_p\alpha e^{(\beta+\theta-N(\xi))(t_1-M)}\right\}$$
$$=\frac{1}{T}\left\{\alpha e^{-\delta t_1}(s\delta+c_1\delta t_1-c\delta-c_1-c_2\delta)+\alpha e^{(\beta+\theta-N(\xi))t_1}(s\beta+sI_e\beta M\right.$$
$$-c(\beta+\theta-N(\xi))-h-d_c[\theta-N(\xi)])$$
$$\left.+\left(\frac{sI_e\beta\left(e^{-(\beta+\theta-N(\xi))M}-1\right)}{\beta+\theta-N(\xi)}-cI_pe^{-(\beta+\theta-N(\xi))M}\right)\alpha e^{(\beta+\theta-N(\xi))t_1}\right\}$$

Proposition II. if $\beta=0$ and $s\delta+c_1t_1\delta-c\delta-c_1-c_2\delta<0$, then $\frac{d^2}{dt^2}P_1(t)<0$ and (t_1^*) is a maximum solution of Equ. (18).

Proof:

$$P_1(t_1)=\frac{1}{T}\left\{s\alpha\left[\frac{t_1[\theta-N(\xi)]}{\beta+\theta-N(\xi)}+\frac{\beta\left(e^{(\beta+\theta-N(\xi))t_1}-1\right)}{[\beta+\theta-N(\xi)]^2}\right]+\frac{s\alpha}{\delta}\left(e^{-\delta t_1}-e^{-\delta T}\right)+sI_e\left[\frac{\alpha M^2\left(\theta-N(\xi)\right)}{2\left(\beta+\theta-N(\xi)\right)}\right.\right.$$
$$\left.+\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}\left(e^{-(\beta+\theta-N(\xi))M}-1\right)}{[\beta+\theta-N(\xi)]^3}+\frac{\beta\alpha Me^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}\right]-K$$
$$-\frac{\alpha h}{\beta+\theta-N(\xi)}\left(\frac{[e^{(\beta+\theta-N(\xi))t_1}-1]}{\beta+\theta-N(\xi)}-t_1\right)-\left(\frac{\alpha c[e^{(\beta+\theta-N(\xi))t_1}-1]}{\beta+\theta-N(\xi)}+\frac{\alpha c}{\delta}\left(e^{-\delta t_1}-e^{-\delta T}\right)\right)$$
$$-\frac{d_c\alpha[\theta-N(\xi)]}{\beta+\theta-N(\xi)}\left[\frac{[e^{(\beta+\theta-N(\xi))t_1}-1]}{\beta+\theta-N(\xi)}-t_1\right]-\frac{c_1\alpha}{\delta}\left[\frac{e^{-\delta T}}{\delta}+Te^{-\delta T}-\frac{e^{-\delta t_1}}{\delta}-t_1e^{-\delta t_1}\right]$$

$$-c_2\alpha\left[t_1+\frac{e^{-\delta t_1}}{\delta}-T-\frac{e^{-\delta T}}{\delta}\right]-\xi T-\frac{cI_p\alpha}{\beta+\theta-N(\xi)}\left[\frac{e^{(\beta+\theta-N(\xi))(t_1-M)}-1}{(\beta+\theta-N(\xi))}+M-t_1\right]\Bigg\}$$

$$=\frac{1}{T}\Bigg\{s\alpha\left[\frac{t_1[\theta-N(\xi)]}{\theta-N(\xi)}\right]+\frac{s\alpha}{\delta}\left(e^{-\delta t_1}-e^{-\delta T}\right)+sI_e\left[\frac{\alpha M^2(\theta-N(\xi))}{2(\theta-N(\xi))}\right]-K$$

$$-\frac{\alpha h}{\theta-N(\xi)}\left(\frac{[e^{(\theta-N(\xi))t_1}-1]}{\theta-N(\xi)}-t_1\right)-\left(\frac{\alpha c[e^{(\theta-N(\xi))t_1}-1]}{\theta-N(\xi)}+\frac{\alpha c}{\delta}\left(e^{-\delta t_1}-e^{-\delta T}\right)\right)$$

$$-\frac{d_c\alpha[\theta-N(\xi)]}{\theta-N(\xi)}\left[\frac{[e^{(\theta-N(\xi))t_1}-1]}{\theta-N(\xi)}-t_1\right]-\frac{c_1\alpha}{\delta}\left[\frac{e^{-\delta T}}{\delta}+Te^{-\delta T}-\frac{e^{-\delta t_1}}{\delta}-t_1e^{-\delta t_1}\right]$$

$$-c_2\alpha\left[t_1+\frac{e^{-\delta t_1}}{\delta}-T-\frac{e^{-\delta T}}{\delta}\right]-\xi T-\frac{cI_p\alpha}{\theta-N(\xi)}\left[\frac{e^{(\theta-N(\xi))(t_1-M)}-1}{(\theta-N(\xi))}+M-t_1\right]\Bigg\}$$

$$=\frac{1}{T}\Bigg\{s\alpha t_1+\frac{s\alpha}{\delta}\left(e^{-\delta t_1}-e^{-\delta T}\right)+sI_e\left[\frac{\alpha M^2}{2}\right]-K-\frac{\alpha h}{\theta-N(\xi)}\left(\frac{[e^{(\theta-N(\xi))t_1}-1]}{\theta-N(\xi)}-t_1\right)$$

$$-\left(\frac{\alpha c[e^{(\theta-N(\xi))t_1}-1]}{\theta-N(\xi)}+\frac{\alpha c}{\delta}\left(e^{-\delta t_1}-e^{-\delta T}\right)\right)$$

$$-d_c\alpha\left[\frac{[e^{(\theta-N(\xi))t_1}-1]}{\theta-N(\xi)}-t_1\right]-\frac{c_1\alpha}{\delta}\left[\frac{e^{-\delta T}}{\delta}+Te^{-\delta T}-\frac{e^{-\delta t_1}}{\delta}-t_1e^{-\delta t_1}\right]$$

$$-c_2\alpha\left[t_1+\frac{e^{-\delta t_1}}{\delta}-T-\frac{e^{-\delta T}}{\delta}\right]-\xi T-\frac{cI_p\alpha}{\theta-N(\xi)}\left[\frac{e^{(\theta-N(\xi))(t_1-M)}-1}{(\theta-N(\xi))}+M-t_1\right]\Bigg\}$$

$$\frac{dP_1(t_1)}{dt_1}=\frac{1}{T}\Bigg\{s\alpha+\frac{s\alpha}{\delta}\left(-\delta e^{-\delta t_1}\right)-\frac{\alpha h}{\theta-N(\xi)}\left(e^{(\theta-N(\xi))t_1}-1\right)-\left(\alpha ce^{(\theta-N(\xi))t_1}+\frac{\alpha c}{\delta}\left(-\delta e^{-\delta t_1}\right)\right)$$

$$-d_c\alpha\left[e^{(\theta-N(\xi))t_1}-1\right]-\frac{c_1\alpha}{\delta}\left[e^{-\delta t_1}-e^{-\delta t_1}+t_1\delta e^{-\delta t_1}\right]-c_2\alpha\left[1-\frac{\delta e^{-\delta t_1}}{\delta}\right]$$

$$-\frac{cI_p\alpha}{\theta-N(\xi)}\left[e^{(\theta-N(\xi))(t_1-M)}-1\right]\Bigg\}$$

$$=\frac{1}{T}\Big\{s\alpha-s\alpha e^{-\delta t_1}-\frac{\alpha h}{\theta-N(\xi)}\left(e^{(\theta-N(\xi))t_1}-1\right)-\left(\alpha ce^{(\theta-N(\xi))t_1}-\alpha ce^{-\delta t_1}\right)$$

$$-d_c\alpha\left[e^{(\theta-N(\xi))t_1}-1\right]-\frac{c_1\alpha}{\delta}\left[t_1\delta e^{-\delta t_1}\right]-c_2\alpha\left[1-e^{-\delta t_1}\right]$$

$$-\frac{cI_p\alpha}{\theta-N(\xi)}\left[e^{(\theta-N(\xi))(t_1-M)}-1\right]\Big\}$$

$$\frac{d^2P_1(t_1)}{dt^2}=\frac{1}{T}\Big\{s\alpha\delta e^{-\delta t_1}-\alpha he^{(\theta-N(\xi))t_1}-\alpha c(\theta-N(\xi))e^{(\theta-N(\xi))t_1}-\alpha c\delta e^{-\delta t_1}$$

$$-d_c\alpha(\theta-N(\xi))e^{(\theta-N(\xi))t_1}-c_1\alpha e^{-\delta t_1}+c_1\alpha t_1\delta e^{-\delta t_1}-c_2\alpha e^{-\delta t_1}$$

$$-cI_p\alpha e^{(\theta-N(\xi))(t_1-M)}\Big\}$$

$$=\frac{1}{T}\Big\{s\alpha\delta e^{-\delta t_1}+c_1\alpha t_1\delta e^{-\delta t_1}-\alpha c\delta e^{-\delta t_1}-c_1\alpha e^{-\delta t_1}-c_2\alpha\delta e^{-\delta t_1}-\alpha he^{(\theta-N(\xi))t_1}$$

$$-\alpha c(\theta-N(\xi))e^{(\theta-N(\xi))t_1}-d_c\alpha(\theta-N(\xi))e^{(\theta-N(\xi))t_1}-cI_p\alpha e^{(\theta-N(\xi))(t_1-M)}\Big\}$$

$$=\frac{1}{T}\Big\{\alpha e^{-\delta t_1}[s\delta+c_1t_1\delta-c\delta-c_1-c_2\delta]-\alpha he^{(\theta-N(\xi))t_1}$$

$$-\alpha c(\theta-N(\xi))e^{(\theta-N(\xi))t_1}-d_c\alpha(\theta-N(\xi))e^{(\theta-N(\xi))t_1}-cI_p\alpha e^{(\theta-N(\xi))(t_1-M)}\Big\}$$

Case II: $t_1\le M$

$$P_2(t_1)=\frac{1}{T}\left\{\begin{matrix}\text{sales revenue}+\text{interest earned}-\text{odering cost}-\text{holding cost}-\text{purchasing cost}\\-\text{deteriorating cost}-\text{shortage cost}-\text{opportunity cost}-\text{preservation cost}\\-\text{interest payable}\end{matrix}\right\}$$

interest payable $=0$

$$\text{interest earned}=sI_e\left[\frac{\alpha t_1^2(\theta-N(\xi))}{2(\beta+\theta-N(\xi))}+\frac{\beta\alpha t_1e^{(\beta+\theta-N(\xi))t_1}}{[\beta+\theta-N(\xi)]^2}-\frac{\beta\alpha[e^{(\beta+\theta-N(\xi))t_1}-1]}{[\beta+\theta-N(\xi)]^3}\right]$$

$$+sI_e(M-t_1)\left[\frac{\alpha t_1[\theta-N(\xi)]}{\beta+\theta-N(\xi)}+\frac{\beta\alpha[e^{(\beta+\theta-N(\xi))t_1}-1]}{[\beta+\theta-N(\xi)]^2}\right]$$

$$P_2(t_1) = \frac{1}{T}\Bigg\{s\alpha\left[\frac{t_1[\theta - N(\xi)]}{\beta + \theta - N(\xi)} + \frac{\beta\left(e^{(\beta+\theta-N(\xi))t_1} - 1\right)}{[\beta + \theta - N(\xi)]^2}\right] + \frac{s\alpha}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right)$$
$$+sI_e\left[\frac{\alpha M^2\left(\theta - N(\xi)\right)}{2\left(\beta + \theta - N(\xi)\right)} + \frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}\left(e^{-(\beta+\theta-N(\xi))M} - 1\right)}{[\beta + \theta - N(\xi)]^3} + \frac{\beta\alpha M e^{(\beta+\theta-N(\xi))t_1}}{[\beta + \theta - N(\xi)]^2}\right] - K$$
$$-\frac{\alpha h}{\beta + \theta - N(\xi)}\left(\frac{[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta + \theta - N(\xi)} - t_1\right) - \left(\frac{\alpha c[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta + \theta - N(\xi)} + \frac{\alpha c}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right)\right)$$
$$-\frac{d_c\alpha[\theta - N(\xi)]}{\beta + \theta - N(\xi)}\left[\frac{[e^{(\beta+\theta-N(\xi))t_1} - 1]}{\beta + \theta - N(\xi)} - t_1\right] - \frac{c_1\alpha}{\delta}\left[\frac{e^{-\delta T}}{\delta} + Te^{-\delta T} - \frac{e^{-\delta t_1}}{\delta} - t_1 e^{-\delta t_1}\right]$$
$$-c_2\alpha\left[t_1 + \frac{e^{-\delta t_1}}{\delta} - T - \frac{e^{-\delta T}}{\delta}\right] - \xi T\Bigg\} \qquad (19)$$

Proposition III. If $\beta = 0,\ s\delta + c_1 t_1 - c\delta - c_1 - c_2\delta < 0,\ e^{-(\beta+\theta-N(\xi))M} \leq 1$ and $s\beta + sI_e\beta M - c(\beta + \theta - N(\xi)) - h - d_c[\theta - N(\xi)] < 0$, then $\frac{d^2}{dt^2}P_1(t) < 0$ and (t_1^*) is a maximum solution of Equ. (19).

Proof:

$$\frac{dP_1(t_1)}{dt_1} = \frac{1}{T}\Bigg\{s\alpha\left[\frac{[\theta - N(\xi)]}{\beta + \theta - N(\xi)} + \frac{\beta e^{(\beta+\theta-N(\xi))t_1}}{\beta + \theta - N(\xi)}\right] + \frac{s\alpha}{\delta}\left(-\delta e^{-\delta t_1}\right)$$
$$+sI_e\left[\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}\left(e^{-(\beta+\theta-N(\xi))M} - 1\right)}{[\beta + \theta - N(\xi)]^2} + \frac{\beta\alpha M e^{(\beta+\theta-N(\xi))t_1}}{\beta + \theta - N(\xi)}\right]$$
$$-\frac{\alpha h}{\beta + \theta - N(\xi)}\left(e^{(\beta+\theta-N(\xi))t_1} - 1\right) - \left(\alpha c e^{(\beta+\theta-N(\xi))t_1} + \frac{\alpha c}{\delta}\left(-\delta e^{-\delta t_1}\right)\right)$$
$$-\frac{d_c\alpha[\theta - N(\xi)]}{\beta + \theta - N(\xi)}\left[e^{(\beta+\theta-N(\xi))t_1} - 1\right] - \frac{c_1\alpha}{\delta}\left[e^{-\delta t_1} - \left(e^{-\delta t_1} - t_1\delta e^{-\delta t_1}\right)\right]$$
$$-c_2\alpha\left[1 - e^{-\delta t_1}\right]\Bigg\}$$
$$= \frac{1}{T}\Bigg\{s\alpha\left[\frac{[\theta - N(\xi)]}{\beta + \theta - N(\xi)} + \frac{\beta e^{(\beta+\theta-N(\xi))t_1}}{\beta + \theta - N(\xi)}\right] - s\alpha e^{-\delta t_1}$$
$$+sI_e\left[\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}\left(e^{-(\beta+\theta-N(\xi))M} - 1\right)}{[\beta + \theta - N(\xi)]^2} + \frac{\beta\alpha M e^{(\beta+\theta-N(\xi))t_1}}{\beta + \theta - N(\xi)}\right]$$
$$-\frac{\alpha h}{\beta + \theta - N(\xi)}\left(e^{(\beta+\theta-N(\xi))t_1} - 1\right) - \left(\alpha c e^{(\beta+\theta-N(\xi))t_1} - \alpha c e^{-\delta t_1}\right)$$
$$-\frac{d_c\alpha[\theta - N(\xi)]}{\beta + \theta - N(\xi)}\left[e^{(\beta+\theta-N(\xi))t_1} - 1\right] - c_1\alpha t_1 e^{-\delta t_1} - c_2\alpha\left[1 - e^{-\delta t_1}\right]\Bigg\}$$

$$\frac{d^2P_1(t_1)}{dt^2} = \frac{1}{T}\Big\{s\alpha\beta e^{(\beta+\theta-N(\xi))t_1} + s\alpha\delta e^{-\delta t_1}$$
$$+sI_e\left[\frac{\beta\alpha e^{(\beta+\theta-N(\xi))t_1}\left(e^{-(\beta+\theta-N(\xi))M} - 1\right)}{\beta + \theta - N(\xi)} + \beta\alpha M e^{(\beta+\theta-N(\xi))t_1}\right]$$
$$-\alpha h e^{(\beta+\theta-N(\xi))t_1} - \alpha c\left(\beta + \theta - N(\xi)\right)e^{(\beta+\theta-N(\xi))t_1} - \alpha c\delta e^{-\delta t_1}$$
$$-d_c\alpha[\theta - N(\xi)]e^{(\beta+\theta-N(\xi))t_1} - c_1\alpha e^{-\delta t_1} + c_1\alpha t_1 e^{-\delta t_1} - c_2\alpha\delta e^{-\delta t_1}\Big\}$$
$$= \frac{1}{T}\Big\{s\alpha\delta e^{-\delta t_1} + c_1\alpha t_1 e^{-\delta t_1} - \alpha c\delta e^{-\delta t_1} - c_1\alpha e^{-\delta t_1} - c_2\alpha\delta e^{-\delta t_1} + s\alpha\beta e^{(\beta+\theta-N(\xi))t_1}$$
$$+sI_e\beta\alpha M e^{(\beta+\theta-N(\xi))t_1} - \alpha c(\beta + \theta - N(\xi))e^{(\beta+\theta-N(\xi))t_1} - \alpha h e^{(\beta+\theta-N(\xi))t_1}$$
$$-d_c\alpha[\theta - N(\xi)]e^{(\beta+\theta-N(\xi))t_1} + \frac{sI_e\beta\alpha e^{(\beta+\theta-N(\xi))t_1}\left(e^{-(\beta+\theta-N(\xi))M} - 1\right)}{\beta + \theta - N(\xi)}\Bigg\}$$
$$= \frac{1}{T}\Big\{\alpha e^{-\delta t_1}(s\delta + c_1 t_1 - c\delta - c_1 - c_2\delta) + \alpha e^{(\beta+\theta-N(\xi))t_1}(s\beta + sI_e\beta M$$
$$-c(\beta + \theta - N(\xi)) - h$$
$$-d_c[\theta - N(\xi)]) + \left(\frac{sI_e\beta\alpha\left(e^{-(\beta+\theta-N(\xi))M} - 1\right)}{\beta + \theta - N(\xi)}\right)e^{(\beta+\theta-N(\xi))t_1}\Bigg\}$$

Proposition IV. if $\beta = 0$ and $\delta s + c_1 t_1 \delta - \delta c - c_1 - c_2\delta < 0$, then $\frac{d^2}{dt^2}P_2(t) < 0$ and (t_1^*) is a maximum solution of Equ. (19).

Proof:

$$P_2(t_1) = \frac{1}{T}\left\{s\alpha\left[\frac{t_1[\theta - N(\xi)]}{\theta - N(\xi)}\right] + \frac{s\alpha}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right) + sI_e\left[\frac{\alpha M^2\left(\theta - N(\xi)\right)}{2\left(\theta - N(\xi)\right)}\right] - K\right.$$
$$-\frac{\alpha h}{\theta - N(\xi)}\left(\frac{[e^{(\theta - N(\xi))t_1} - 1]}{\theta - N(\xi)} - t_1\right) - \left(\frac{\alpha c[e^{(\theta - N(\xi))t_1} - 1]}{\theta - N(\xi)} + \frac{\alpha c}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right)\right)$$
$$-\frac{d_c\alpha[\theta - N(\xi)]}{\theta - N(\xi)}\left[\frac{[e^{(\theta - N(\xi))t_1} - 1]}{\theta - N(\xi)} - t_1\right] - \frac{c_1\alpha}{\delta}\left[\frac{e^{-\delta T}}{\delta} + Te^{-\delta T} - \frac{e^{-\delta t_1}}{\delta} - t_1 e^{-\delta t_1}\right]$$
$$\left.-c_2\alpha\left[t_1 + \frac{e^{-\delta t_1}}{\delta} - T - \frac{e^{-\delta T}}{\delta}\right] - \xi T\right\}$$
$$= \frac{1}{T}\left\{s\alpha t_1 + \frac{s\alpha}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right) + sI_e\left[\frac{\alpha M^2}{2}\right] - K - \frac{\alpha h}{\theta - N(\xi)}\left(\frac{[e^{(\theta - N(\xi))t_1} - 1]}{\theta - N(\xi)} - t_1\right)\right.$$
$$-\left(\frac{\alpha c[e^{(\theta - N(\xi))t_1} - 1]}{\theta - N(\xi)} + \frac{\alpha c}{\delta}\left(e^{-\delta t_1} - e^{-\delta T}\right)\right) - d_c\alpha\left[\frac{[e^{(\theta - N(\xi))t_1} - 1]}{\theta - N(\xi)} - t_1\right]$$
$$\left.-\frac{c_1\alpha}{\delta}\left[\frac{e^{-\delta T}}{\delta} + Te^{-\delta T} - \frac{e^{-\delta t_1}}{\delta} - t_1 e^{-\delta t_1}\right] - c_2\alpha\left[t_1 + \frac{e^{-\delta t_1}}{\delta} - T - \frac{e^{-\delta T}}{\delta}\right] - \xi T\right\}$$

$$\frac{dP_1(t_1)}{dt_1} = \frac{1}{T}\left\{s\alpha + \frac{s\alpha}{\delta}\left(-\delta e^{-\delta t_1}\right) - \frac{\alpha h}{\theta - N(\xi)}\left(e^{(\theta - N(\xi))t_1} - 1\right) - \left(\alpha c e^{(\theta - N(\xi))t_1} + \frac{\alpha c}{\delta}\left(-\delta e^{-\delta t_1}\right)\right)\right.$$
$$\left.-d_c\alpha[e^{(\theta - N(\xi))t_1} - 1] - \frac{c_1\alpha}{\delta}[e^{-\delta t_1} - e^{-\delta t_1} + t_1\delta e^{-\delta t_1}] - c_2\alpha[1 - e^{-\delta t_1}]\right\}$$
$$= \frac{1}{T}\left\{s\alpha - s\alpha e^{-\delta t_1} - \frac{\alpha h}{\theta - N(\xi)}\left(e^{(\theta - N(\xi))t_1} - 1\right) - \left(\alpha c e^{(\theta - N(\xi))t_1} - \alpha c e^{-\delta t_1}\right)\right.$$
$$\left.-d_c\alpha[e^{(\theta - N(\xi))t_1} - 1] - \frac{c_1\alpha}{\delta}[t_1\delta e^{-\delta t_1}] - c_2\alpha[1 - e^{-\delta t_1}]\right\}$$

$$\frac{d^2P_1(t_1)}{dt^2} = \frac{1}{T}\left\{\delta s\alpha e^{-\delta t_1} - \frac{\alpha h}{\theta - N(\xi)}\left(\left(\theta - N(\xi)\right)e^{(\theta - N(\xi))t_1}\right)\right.$$
$$-\left(\alpha c\left(\theta - N(\xi)\right)e^{(\theta - N(\xi))t_1} + \delta\alpha c e^{-\delta t_1}\right) - d_c\alpha\left[\left(\theta - N(\xi)\right)e^{(\theta - N(\xi))t_1}\right]$$
$$\left.-\frac{c_1\alpha}{\delta}[\delta e^{-\delta t_1} - t_1\delta^2 e^{-\delta t_1}] - c_2\alpha[\delta e^{-\delta t_1}]\right\}$$
$$= \frac{1}{T}\left\{\delta s\alpha e^{-\delta t_1} - \alpha h e^{(\theta - N(\xi))t_1} - \alpha c\left(\theta - N(\xi)\right)e^{(\theta - N(\xi))t_1} - \delta\alpha c e^{-\delta t_1}\right.$$
$$\left.-d_c\alpha\left(\theta - N(\xi)\right)e^{(\theta - N(\xi))t_1} - c_1\alpha e^{-\delta t_1} + c_1\alpha t_1\delta e^{-\delta t_1} - c_2\alpha\delta e^{-\delta t_1}\right\}$$
$$= \frac{1}{T}\left\{\delta s\alpha e^{-\delta t_1} + c_1\alpha t_1\delta e^{-\delta t_1} - \delta\alpha c e^{-\delta t_1} - c_1\alpha e^{-\delta t_1} - c_2\alpha\delta e^{-\delta t_1} - \alpha h e^{(\theta - N(\xi))t_1}\right.$$
$$\left.-\alpha c\left(\theta - N(\xi)\right)e^{(\theta - N(\xi))t_1} - d_c\alpha\left(\theta - N(\xi)\right)e^{(\theta - N(\xi))t_1}\right\}$$
$$= \frac{1}{T}\left\{\alpha e^{-\delta t_1}(\delta s + c_1 t_1\delta - \delta c - c_1 - c_2\delta) - \alpha h e^{(\theta - N(\xi))t_1}\right.$$
$$\left.-\alpha c\left(\theta - N(\xi)\right)e^{(\theta - N(\xi))t_1} - d_c\alpha\left(\theta - N(\xi)\right)e^{(\theta - N(\xi))t_1}\right\}$$

Numerical Analysis and Sensitivity Analysis

Case I: $M \leq t_1$

For numerical examples, we solved Equ. (18)., an inventory system with the following parameter set. α=1000units, β=0.6, T=20days, M=10days, c=1Naira/unit, h=4Naira/unit time, c_1=1Naira/unit/order, c_2=1Naira/unit/order, θ=0.7, K=2000, N=0.06, ξ=3, d_c=0.4, I_p=0.14, $\sigma = 0.03$, $s = 1.2$Naira
I_e=0.14.
The optimal time $t_1^* = 18.1550$
Optimal Total average cost $P_1^* = 2862$
Optimal order quantity $Q^* = 48747$

Table 1. The effect of the parameter α on P_1^* and Q^*

α	P_1^*	Q^*
1000	2862	48747
2000	5621	97493
3000	8381	146240
4000	11141	194990
5000	13900	243730

β=0.04, T=40days, M=30days, c=1Naira/unit, h=1Naira/unit time, c_1=1Naira/unit/order, c_2=1Naira/unit/order, θ=0.8, K=30, N=0.02, ξ=0.3, d_c=0.3, $\sigma = 0.03$ I_p=0.14.

Table 2. The effect of the parameter N on P_1^* and Q^*

N	P_1^*	Q^*
0.04	2891	48835
0.05	2876	48789
0.06	2862	48747
0.07	2848	48707
0.08	2834	48671
0.09	2820	48638

α=40units, T=40days, M=30days, c=1Naira/unit, c_1=1Naira/unit/order,
h=1Naira/unit time, c_2=1Naira/unit/order, θ=0.8, K=30, N=0.02, ξ=0.3, d_c=0.3, I_p=0.14, $\sigma = 0.03$.

Case II: $t_1 \leq M$

For numerical examples, we solved Equ. (19)., an inventory system with the following parameter set. α=1000units, β=0.6, T=11days, M=10days, c=1Naira/unit, h=4Naira/unit time, c_1=1Naira/unit/order, c_2=1Naira/unit/order, θ=0.7, K=2000, N=0.06, ξ=3, d_c=0.4, I_p=0.14, $\sigma = 0.03$, $s = 1.2$Naira
I_e=0.14,
The optimal time $t_1^* = 8.7672$
Optimal Total average cost $P_2^* = 3056$
Optimal order quantity $Q^* = 21966$

Table 3. The effect of the parameter α on P_2^* and Q^*

α	P_2^*	Q^*
1000	3056	21966
2000	5927	43931
3000	8798	65897
4000	11670	87863
5000	14541	109830

β=0.04, T=40days, c=1Naira/unit, h=1Naira/unit time, c_2=1Naira/unit/order,
c_1=1Naira/unit/order,θ=0.7, K=30, N=0.02, ξ=0.3, d_c=0.3, $\sigma = 0.03$.

Table 4. The effect of the parameter N on P_2^* and Q^*

N	P_2^*	Q^*
0.04	3073	22048
0.05	3064	22006
0.06	3056	21966
0.07	3047	21925
0.08	3039	21886
0.09	3032	21849

α=40units, T=40days, c=1Naira/unit, h=1Naira/unit time, c_1=1Naira/unit/order, c_2=1Naira/unit/order, θ=0.7, K=30, N=0.02, ξ=0.3, d_c=0.3, $\sigma = 0.03$.

From Table 1 and Table 2, it can be observed that the optimal profit and optimal quantity are sensitive to the demand and reduced deterioration rate parameters. From Table 1, it can be seen that as the demand increases, the optimal profit and optimal quantity also increased. Therefore, the optimal decision is to increase the quantity purchased as demand increases. Also from Table 2, it can be observed that as reduced deterioration rate increases, the optimal profit and optimal quantity decreases. Hence, the optimal decision is to keep the reduced deterioration rate as low as possible in other to maximized profit. Furthermore, from Table 3 and Table 4, it can be seen that optimal profit and optimal quantity are also sensitive to demand and reduced deterioration rates parameters. Again from Table 3, it can be observed that as the demand increases, the optimal profit and optimal quantity also increased. Again, the optimal decision is to increase the quantity purchased as demand increases. Also as obtained in Table 4, as reduced deterioration rate increases, the optimal profit and optimal quantity decreases. So, the optimal decision is to keep the reduced deterioration rate as low as possible in other to maximized profit.

Conclusion

This work studied the optimal inventory policies for inventory model for managing deteriorating product with stock level dependent demand and controllable deterioration under trade credit and partial backlogging by maximizing the derived objective functions for the case where retailer's trade credit period offered by supplier is less than the time at which the inventory level falls to zero and the case where the time at which the inventory level falls to zero is less than the retailer's trade credit period offered by supplier. We also presented numerical illustration and sensitivity analysis to illustrate the obtained results for the two cases.

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